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Entropy of Dilatonic Black Hole

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By using the method of quantum statistics, we directly derive the partition function of bosonic and fermionic field in dilatonic black hole and obtain the integral expression of the black hole's entropy, which avoids the difficulty in solving the wave equation of various particles. Then via the improved brick-wall method, membrane model, we obtain that we can choose proper parameter in order to let the thickness of film tend to zero and have it approach the surface of its horizon. Consequently the entropy of the black hole is proportional to the area of its horizon. In our result, the stripped term and the divergent logarithmic term in the original brick-wall method no longer exist. In the whole process, physics idea is clear; calculation is simple. We offer a new simple and direct way of calculating the entropy of different complicated black holes.

KEY WORDS: membrane model; entropy of a black hole; quantum statistics.

1. INTRODUCTION

The entropy of a black hole is one of the important subjects in theoretical physics. Because entropy has statistical meaning, understanding the entropy of the black hole involves the microscopic essence of the black hole. Because Bekenstein and Hawking put forward that the entropy of a black hole is proportional to the area of its event horizon (Bekenstein, 1973; Gibbons and Hawking, 1977a; Hawking, 1975), the statistical origin of the black hole has been probed and many ways of calculating entropy have emerged as times required (Cai *et al.*, 1998; Frolov *et al.*, 1996; Lee *et al.*, 1996; Mann and Solodukhin, 1996; Srednicki, 1993; 't Hooft, 1985). The most frequently used method is the brick-wall method advanced by 't Hooft (1985). This method is used to study the statistical properties of a free scalar field in an asymptotically flat space-time in various spherical coordinates (Jing, 1995; Lee and Kim, 1996; Zhao *et al.*, 2000) and it is found that the general expression of the black hole's entropy consists of a term which is proportional to the area of its event horizon and a divergent logarithmic term which is not

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proportional to the area of its event horizon. However it is doubted that, first, the entropy of the scalar or Dirac field outside the event horizon is the entropy of the black hole; second, the state density near the event horizon is divergent; third, whether the logarithmic term and L^3 term belong to the entropy of the black hole; fourth, why the approximation of small mass is taken to obtain the entropy of the scalar or Dirac field. The above mentioned problems with the original brick-wall method are unnatural.

We derive the bosonic and fermionic partition functions in Dilatonic black hole directly by using the quantum statistical method and then obtain the integral expression of the system's entropy. Then we use the membrane model to calculate entropy (Zhao *et al.*, 2001). As a result, the left out term in original brick-wall method no longer exists. The problem that the state density near the event horizon is divergent doesn't exist either. We also consider the spinning degeneracy of radiational particles. In the whole process, the physics idea is clear, calculation is simple and the result is reasonable. It offers a neat way of studying the black hole's entropy. In this article, we take the simplest functional form of the temperature ($C = \hbar = K_{\mathrm{B}} = 1$).

2. BOSONIC ENTROPY

The linear element of space-time in Dilatonic black hole is given by Ghosh and Mitra (1994):

$$dS^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r(r-a)\, d\Omega^2, \qquad (2.1)$$

where $a = \frac{Q^2}{2M} e^{-2\phi_0}$, M and Q are mass and charges and ϕ_0 is a constant.

Hawking radiation temperature of the black hole is as follows:

$$T_{\mathrm{H}} = \frac{1}{8\pi M}. \qquad (2.2)$$

The location of horizon is $r_+ = 2M$ and the area of horizon is as follows:

$$A_{\mathrm{H}} = 4\pi 2M(2M - a). \qquad (2.3)$$

Based on the theory of general relativity, an observer at rest at an infinite distance gets the frequency shift of the particles from the surface of a star as follows:

$$\nu = \nu_0 \chi, \qquad (2.4)$$

where ν_0 is the natural frequency of the atoms on the surface of star and ν is the one obtained by the observer at rest at an infinite distance.

In the view of Tolman (1934), the natural radiation temperature got by the observer at rest at an infinite distance is as follows:

$$T = \frac{T_{\mathrm{H}}}{\chi}, \tag{2.5}$$

where $\chi = \sqrt{1 - \frac{2M}{r}}$ is the red-shift factor.

For bosonic gas, we calculate the grand partition function as follows:

$$\ln Z = -\sum_i g_i \ln(1 - e^{-\beta\varepsilon_i}). \tag{2.6}$$

In unit volume, the number of quantum states with the energy between ε and $\varepsilon + d\varepsilon$ or the frequency between ν and $\nu + d\nu$ is as follows:

$$g(\nu)\, d\nu = j 4\pi \nu^2\, d\nu, \tag{2.7}$$

where j is the spinning degeneracy of the particles. Because in space-time (2.1) the area of two-dimensional curved surface at random point r is $4\pi r(r-a)$, the partition function of system in lamella with random thickness near the outside of horizon is as follows:

$$\begin{aligned}\ln Z &= \int 4\pi r(r-a)\frac{dr}{\chi}\sum_i g_i \sum_{n=1} \frac{1}{n}\, e^{-n\beta\varepsilon_i}\\ &= j16\pi^2 \int r(r-a)\frac{dr}{\chi}\sum_{n=1}\frac{1}{n}\int_0^\infty e^{-\frac{nh\nu}{T}}\nu^2\, d\nu\\ &= j\frac{2\pi^3}{45}\int r(r-a)\frac{dr}{\beta^3\chi},\end{aligned} \tag{2.8}$$

where $\frac{1}{\beta} = T$. Using the relation between entropy and partition function

$$S = \ln Z - \beta_0 \frac{\partial \ln Z}{\partial \beta_0}, \tag{2.9}$$

we have:

$$S_b = j\frac{8\pi^3}{45}\frac{1}{\beta_0^3}\int r(r-a)\frac{dr}{\chi^4}, \tag{2.10}$$

where $\beta = \beta_0\chi$, and $\beta_0 = \frac{1}{T_{\mathrm{H}}}$. In the above integral (2.10), we take the integral region $[r_+ + \varsigma,\ r_+ + N\varsigma]$, where ς is a small nonnegative quantity and N is a constant larger than one. So (2.10) can be written as:

$$S_b = j\frac{8\pi^3}{45}\frac{1}{\beta_0^3}\int_{r_++\varsigma}^{r_++N\varsigma}\frac{r^3(r-a)\,dr}{(r-2M)^2}$$
$$= j\frac{8\pi^3}{45}\frac{1}{\beta_0^3}r_+^3(r_+-a)\left[\frac{N-1}{N\varsigma}\right]+F(N,\varsigma), \tag{2.11}$$

where

$$F(N,\varsigma) = j\frac{8\pi^3}{45}\frac{1}{\beta_0^3}\int_{r_++\varsigma}^{r_++N\varsigma}\left[\frac{4r_+^3-3ar_+^2}{r-r_+}+(6r_+^2-3ar_+)\right.$$
$$\left.+(4r_+-a)(r-r_+)+(r-r_+)^2\right]dr$$
$$= j\frac{8\pi^3}{45}\frac{1}{\beta_0^3}\left[(4r_+^3-3ar_+^2)\ln N+(6r_+^2-3ar_+)\varsigma(N-1)\right.$$
$$\left.+\frac{1}{2}(4r_+-a)\varsigma^2(N^2-1)+\frac{1}{3}\varsigma^3(N^3-1)\right].$$

From (3.17) in 't Hooft (1985), we know that when $N_\varsigma = L \gg r_+$ (that is, $N \gg 1$), if we take $\varsigma = \frac{T_H}{90}$ as ultraviolet cutoff, the main part of the black hole's entropy is proportional to the area of horizon, which is the result of 't Hooft brick-wall.

Now we use membrane model for discussion. In (2.11), N and ς are two independent parameters. We take N to be slightly bigger than one instead of $N \gg 1$. So the integration in (2.11) is only done within the film with the thickness of $(N-1)\varsigma$. We take

$$\varsigma = \frac{T_H}{90}\frac{N-1}{N}. \tag{2.12}$$

So the black hole's entropy can be expressed as follows:

$$S_b = j\pi r_+(r_+-a)+F(N,\varsigma) = j\frac{1}{4}A_H+F(N,\varsigma). \tag{2.13}$$

As $N \to 1$, $\varsigma \to 0$, and $N\varsigma \to 0$, that is, the integral upper limit and lower limit both tend to the outer horizon. The thickness of film is zero and keeps close to the surface of horizon. In other words as $N \to 1$, the extreme of film is horizon. Because $\lim_{N\to 1} F(N,\varsigma) \to 0$, the black hole's entropy is as follows:

$$S_b = \frac{j}{4}A_H. \tag{2.14}$$

Because we let the integral upper limit and lower limit both tend to the outer horizon, the entropy obtained in (2.14) is independent of the radiation field outside horizon. It only has the property of two-dimensional membrane in three-dimensional space. So the obtained entropy has the property of two-dimensional membrane. The existence of horizon is the basic property of the black hole. It has already been proved that the general existence of horizon leads to the Hawking effect (Zhao, 1981). And whether there is the black hole's entropy or not directly involves the existence of horizon (Gibbons and Hawking, 1977b). So the entropy in (2.14) should be the black hole's entropy. When $j = 1$ for radiation particles, we obtain that the black hole's entropy is a quarter of the area of horizon. When $j \neq 1$, we can take j into consideration in the parameters N and ς in (2.12) to make sure that the black hole's entropy is a quarter of the area of horizon.

3. FERMIONIC ENTROPY

For Fermionic gas, the grand partition function is as follows:

$$\ln Z = \sum_i g_i \ln(1 + e^{-\beta\varepsilon_i}). \tag{3.1}$$

From (2.7), we obtain

$$\begin{aligned} \ln Z &= \int 4\pi r(r-a)\frac{dr}{\chi}\sum_i g_i \sum_{n=1} \frac{(-1)^{n-1}}{n}\, e^{-n\beta\varepsilon_i} \\ &= i16\pi^2 \int r(r-a)\frac{dr}{\chi}\sum_{n=1}\frac{(-1)^{n-1}}{n}\int_0^\infty e^{-\frac{nh\nu}{T}}\nu^2\, d\nu \\ &= i\frac{2\pi^3}{45}\frac{7}{8}\int r(r-a)\frac{dr}{\beta^3\chi}. \end{aligned} \tag{3.2}$$

Using the result of part 2, we can get the fermionic entropy of Dilatonic black hole as follows:

$$S_f = i\frac{1}{4}\frac{7}{8}A_{\mathrm{H}}, \tag{3.3}$$

where i is spinning degeneracy of fermionic particles. In fact, $\frac{7}{8}i$ in (3.3) can also be taken into consideration in (2.12).

4. CONCLUSION

In the above analysis, we derive partition functions of various fields in Dilatonic black hole directly by using the statistical method. We avoid the difficulty in solving wave equation. Because we use the improved brick-wall method, membrane model, to calculate the entropy of various fields, the problem that the state density is divergent around horizon does not exist any more. In our calculation, as $N \to 1$, $\varsigma \to 0$, and $N\varsigma \to 0$, that is, the inner and outer "brick walls" both approach the outer horizon of the black hole. From (2.14) to (3.3), we know that the divergent logarithmic term and L^3 term in the original brick-wall method no longer exist. The obtained entropy is proportional to the area of its horizon, so it can be taken as the black hole's entropy.

In above analysis, we know that by using the statistical and membrane model methods, the doubt that the entropy of the scalar or Dirac field outside the event horizon is the entropy of the black hole in the original brick-wall method doesn't exist and the complicated approximations in solution is avoided. In the whole process, the physics idea is clear; the calculation is simple; and the result is reasonable. We also consider the influence of the spinning degeneracy of particles on the entropy. For calculating the entropy in various space-times, we only need to change the red-shift factor, but the others are the same. Especially for complicated space-times, we can directly derive the entropy of various quantum particles without solving the complicated wave equation. We offer a new neat way of studying the entropy of different kinds of complicated black holes.

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